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PORTFOLIO SELECTION WITH REGIME-SWITCHING: DYNAMIC PROGRAMMING APPROACHES

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Abstract. I study an optimal consumption and portfolio selection problem with regime-switching using a dynamic programming method. With constant relative risk aversion (CRRA) utility I obtain optimal solutions in closed-form.

1. Introduction

Motivated by Merton's seminal works [5, 6], much scholarly work has been conducted on portfolio optimization problems. Especially Karatzas *et al.* [4] have derived general explicit solutions of a consumption / investment optimization problem using a dynamic programming method. The regime-switching model, in which uncertainty is affected not only by the Brownian motion of a stock, but also by the long-term business cycle which is mathematically modeled as a continuous-time Markov chain, is becoming widely considered in mathematical finance (option pricing [2], real options [1], portfolio selection problems [3, 7]).

In this paper I investigate an optimal consumption and portfolio selection problem with regime-switching under the framework of Karatzas *et al.* [4]. With constant relative risk aversion (CRRA) utility I derive optimal solutions in closed-form.

2. The financial market

For a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, I define a standard Brownian motion B_t and a continuous-time two-state Markov chain ϵ_t. It is assumed that B_t and ϵ_t are independent and the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is generated by the Brownian motion B_t and the Markov chain ϵ_t.

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In this financial market, two assets are traded: One is a money market account and the other is a stock. It is assumed that there are two regimes 1, 2 in the market and regime i switches into regime j at the first jump time of an independent Poisson process with intensity λ_i, for $i, j \in \{1, 2\}$. So, in regime $i \in \{1, 2\}$, the interest rate is r_i, and the stock price process is governed by $dS_t/S_t = \mu_i dt + \sigma_i dB_t$. The market price of risk is defined by $\theta_i \triangleq \frac{\mu_i - r_i}{\sigma_i}$ $i = 1,\ 2$. Let π_t be the $\mathcal{F}_t$-progressively measurable portfolio process, the amount of the agent's wealth invested in the risky asset at time t and c_t be the nonnegative $\mathcal{F}_t$-progressively measurable consumption rate process at time t. It is assumed that they satisfy the following conditions:

$$\int_0^t c_s ds < \infty \quad \text{and} \quad \int_0^t \pi_s^2 ds < \infty, \ \text{for all } t \geq 0, \ \text{almost surely (a.s.)}.$$

The agent's wealth process X_t at time t follows the stochastic differential equation (SDE)

$$dX_t = [r_i X_t + \pi_t(\mu_i - r_i) - c_t]\, dt + \sigma_i \pi_t dB_t, \quad X_0 = x > 0, \ i = 1, \ 2.$$

3. The optimization problem

The agent's optimization problem is to maximize her expected utility

$$V_i(x) = \sup_{(c,\pi)\in\mathcal{A}(x)} \mathbb{E}\left[\int_0^{T_i} e^{-\beta t}\frac{c_t^{1-\gamma}}{1-\gamma}dt + e^{-\beta T_i}V_j(X_{T_i})\right], \tag{3.1}$$

where T_i is the first jump time from i-th state to j-th state, $\beta > 0$ is a subjective discount factor, and $\mathcal{A}(x)$ is an admissible class of pair (c, π) at x, where $i, j \in \{1, 2\}$ and $i \neq j$.

In order to guarantee the agent's problem to be well-defined, we have the following assumption.

Assumption 1.

$$K_i \triangleq r_i + \frac{\beta - r_i}{\gamma} + \frac{\gamma - 1}{2\gamma^2}\theta_i^2 > 0, \ i = 1, \ 2.$$

The next theorem gives my main results.

Theorem 1. *The value function is given by*

$$V_i(x) = M_i \frac{x^{1-\gamma}}{1-\gamma},$$

where M_i and M_j are the solutions to the system of algebraic equations

$$-(\gamma K_i + \lambda_i)M_i + \gamma M_i^{-\frac{1-\gamma}{\gamma}} + \lambda_i M_j = 0,$$

where $i, j \in \{1, 2\}$ and $i \neq j$. And the optimal strategies are given by (c_i^*, π_i^*) such that

$$c_i^* = M_i^{-\frac{1}{\gamma}} x \quad \text{and} \quad \pi_i^* = \frac{\theta_i}{\gamma\sigma_i} x, \ i = 1, \ 2.$$

Proof. From the optimization problem (3.1), I derive the Bellman equations as follows:

$$\begin{aligned} \max_{(c_i,\pi_i)} \Bigg[& \{r_i x + \pi_i(\mu_i - r_i) - c_i\} V_i'(x) + \frac{1}{2}\sigma_i^2 \pi_i^2 V_i''(x) \\ & -(\beta + \lambda_i)V_i(x) + \lambda_i V_j(x) + \frac{c_i^{1-\gamma}}{1-\gamma} \Bigg] = 0, \end{aligned} \tag{3.2}$$

where $i, j \in \{1, 2\}$ and $i \neq j$. By the first-order conditions (FOCs), I derive

$$c_i = (V_i'(x))^{-\frac{1}{\gamma}} \quad \text{and} \quad \pi_i = -\frac{\theta_i V_i'(x)}{\sigma_i V_i''(x)}, \ i = 1, \ 2. \tag{3.3}$$

I now assume that the optimal consumption $c_i^* = C_i(x), \ i = 1, \ 2$, is a function of wealth and that $X_i(\cdot)$ is the inverse function of $C_i(\cdot)$. Then, from the FOCs in (3.3), I have

$$V_i'(x) = C_i(x)^{-\gamma} \quad \text{and} \quad V_i''(x) = \frac{-\gamma C_i(x)^{-\gamma-1}}{X_i'(c_i)}, \ i = 1, \ 2. \tag{3.4}$$

Substituting the FOCs in (3.3) with (3.4) into the equation (3.2), then

$$\begin{aligned} & r_i c_i^{-\gamma} X_i(c_i) + \frac{\theta_i^2}{2\gamma} c_i^{1-\gamma} X_i'(c_i) - (\beta + \lambda_i)V_i(X_i(c_i)) \\ & + \lambda_i V_j(X_i(c_i)) + \frac{\gamma}{1-\gamma} c_i^{1-\gamma} = 0, \end{aligned} \tag{3.5}$$

where $i, j \in \{1, 2\}$ and $i \neq j$. Differentiating the equation (3.5) with respect to c_i, then

$$\begin{aligned} & r_i c_i^{-\gamma} X_i'(c_i) - r_i \gamma c_i^{-\gamma-1} X_i(c_i) + \frac{\theta_i^2}{2\gamma} c_i^{1-\gamma} X_i''(c_i) \\ & + \frac{1-\gamma}{2\gamma} \theta_i^2 c_i^{-\gamma} X_i'(c_i) - (\beta + \lambda_i) c_i^{-\gamma} X_i'(c_i) \\ & + \lambda_i V_j'(X_i(c_i)) X_i'(c_i) + \gamma c_i^{-\gamma} = 0, \end{aligned} \tag{3.6}$$

where $i, j \in \{1,2\}$ and $i \neq j$. From the definition of $C_i(\cdot)$ and $X_i(\cdot)$, I see that

$$x = X_1(c_1) = X_2(c_2) \quad \text{implies} \quad V_j'(X_i(c_i)) = V_j'(x) = c_j^{-\gamma}, \tag{3.7}$$

where $i, j \in \{1,2\}$ and $i \neq j$. From the equation (3.6), I derive

$$\begin{aligned} \frac{\theta_i^2}{2\gamma} c_i^2 X_i''(c_i) - \left(\lambda_i + \beta - r_i - \frac{1-\gamma}{2\gamma}\theta_i^2\right) c_i X_i'(c_i) - r_i \gamma X_i(c_i) \\ + \lambda_i c_i \left(\frac{c_i}{c_j}\right)^{\gamma} X_i'(c_i) + \gamma c_i = 0, \end{aligned} \tag{3.8}$$

where $i, j \in \{1,2\}$ and $i \neq j$. If I conjecture the solution $X_i(c_i)$ of the form

$$X_i(c_i) = M_i^{\frac{1}{\gamma}} c_i \quad \text{and} \quad c_i = M_i^{-\frac{1}{\gamma}} x, \; i = 1, \; 2, \tag{3.9}$$

then $X_i'(c_i) = M_i^{\frac{1}{\gamma}}$ and $X_i''(c_i) = 0, \; i = 1, \; 2$. From the equality (3.7), I obtain

$$1 = \frac{X_i(c_i)}{X_j(c_j)} = \frac{M_i^{\frac{1}{\gamma}} c_i}{M_j^{\frac{1}{\gamma}} c_j} \quad \text{and} \quad \left(\frac{c_i}{c_j}\right)^{\gamma} = \frac{M_j}{M_i},$$

where $i, j \in \{1,2\}$ and $i \neq j$. So the equation (3.8) can be reduced into

$$-(\gamma K_i + \lambda_i) M_i + \gamma M_i^{-\frac{1-\gamma}{\gamma}} + \lambda_i M_j = 0, \tag{3.10}$$

where $i, j \in \{1,2\}$ and $i \neq j$. I can easily show that there is a unique pair solution (M_1, M_2) to (3.10) using the proof of Lemma 1 of Jang *et al.* [3] under Assumption 1. I also derive the value function $V_i(x) = M_i \frac{x^{1-\gamma}}{1-\gamma}$, by substituting c_i in (3.9) into the FOCs in (3.3) and the Bellman equation (3.2). Finally I use the FOCs in (3.3) with the value function $V_i(\cdot)$ to obtain the optimal policies $(c_i^*, \pi_i^*), \; i = 1, \; 2$. □

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